UHLENBECK COMPACTNESS AND TRANSVERSALITY FOR THE MODULI SPACE OF PU(2) MONOPOLES[1]

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1. Introduction

This research announcement reports on some results in [11] concerning the existence of perturbations for the PU(2) monopole equations, yielding both useful transversality properties and an Uhlenbeck compactification for this perturbed moduli space.

A method has been proposed in [24, 25, 27] to prove Witten's conjecture concerning the relation between the Donaldson and Seiberg-Witten invariants of smooth four-manifolds [6, 40, 21]. The idea is to use a moduli space of solutions to the PU(2) monopole equations, which are a natural generalization of the U(1) monopole equations of Seiberg and Witten and the anti-self-dual equation for SO(3) connections, to construct a cobordism between links of compact moduli spaces of U(1) monopoles of Seiberg-Witten type and the moduli space of anti-self-dual connections, which appear as singularities in this larger moduli space. The conjecture holds for all four-manifolds whose Donaldson and Seiberg-Witten invariants have been independently computed by direct calculation. A basic requirement of this cobordism technique is the existence of an Uhlenbeck compactification for the moduli space of PU(2) monopoles and of generic-parameter transversality results for all the moduli spaces of PU(2) monopoles which appear in this compactification, at least away from the anti-self-dual and U(1) solutions.

In §2 we describe the holonomy perturbations we use in order to achieve transversality for the moduli space of solutions to the perturbed PU(2) monopole equations. In §3 and §4 we state our Uhlenbeck compactness and transversality theorems for this moduli space and outline the proofs from [11], to which we refer for detailed arguments.

2. Holonomy perturbations

We consider Hermitian two-plane bundles E over X whose determinant line bundles $\det E$ are isomorphic to a fixed Hermitian line bundle over X endowed with a fixed C^∞, unitary connection A_e. Let (ρ, W) be a spin^c structure on X, where $\rho : T^*X \to \operatorname{End} W$ is the Clifford map, and the Hermitian four-plane bundle $W =$

$W^+ \oplus W^-$ is endowed with a C^∞ spin^c connection. *The spin^c structure (ρ, W), the spin^c connection on W, and the Hermitian line bundle together with its connection A_e are fixed once and for all.*

Let $k \geq 2$ be an integer and let $\mathcal{A}_E$ be the space of L_k^2 connections A on the U(2) bundle E all inducing the fixed determinant connection on $\det E$. Equivalently, following [19, §2(i)], we may view $\mathcal{A}_E$ as the space of L_k^2 connections A on the PU(2) = SO(3) bundle $\mathfrak{su}(E)$. We shall often pass back and forth between these viewpoints, via the fixed connection on $\det E$, relying on the context to make the distinction clear. Let $D_A : L_k^2(W^+ \otimes E) \to L_{k-1}^2(W^- \otimes E)$ be the corresponding Dirac operators. Given a connection A on E with curvature $F_A \in L_{k-1}^2(\Lambda^2 \otimes \mathfrak{u}(E))$, then $(F_A^+)_0 \in L_{k-1}^2(\Lambda^+ \otimes \mathfrak{su}(E))$ denotes the traceless part of its self-dual component. Equivalently, if A is a connection on $\mathfrak{su}(E)$ with curvature $F_A \in L_{k-1}^2(\Lambda^2 \otimes \mathfrak{so}(\mathfrak{su}(E)))$, then $\mathrm{ad}^{-1}(F_A^+) \in L_{k-1}^2(\Lambda^+ \otimes \mathfrak{su}(E))$ is its self-dual component, viewed as a section of $\Lambda^+ \otimes \mathfrak{su}(E)$ via the isomorphism $\mathrm{ad} : \mathfrak{su}(E) \to \mathfrak{so}(\mathfrak{su}(E))$. When no confusion can arise, the isomorphism $\mathrm{ad} : \mathfrak{su}(E) \to \mathfrak{so}(\mathfrak{su}(E))$ will be implicit and so we regard F_A as a section of $\Lambda^+ \otimes \mathfrak{su}(E)$ when A is a connection on $\mathfrak{su}(E)$.

For an L_k^2 section Φ of $W^+ \otimes E$, let Φ^* be its pointwise Hermitian dual and let $(\Phi \otimes \Phi^*)_{00}$ be the component of the Hermitian endomorphism $\Phi \otimes \Phi^*$ of $W^+ \otimes E$ which lies in $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$. The Clifford multiplication ρ defines an isomorphism $\rho^+ : \Lambda^+ \to \mathfrak{su}(W^+)$ and thus an isomorphism $\rho^+ = \rho^+ \otimes \mathrm{id}_{\mathfrak{su}(E)}$ of $\Lambda^+ \otimes \mathfrak{su}(E)$ with $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$. Then

$$(2.1) \qquad F_A^+ - (\rho^+)^{-1}(\Phi \otimes \Phi^*)_{00} = 0,$$
$$D_A \Phi = 0,$$

are the unperturbed equations considered in [23, 24, 25, 27] (the trace conditions vary slightly — see [11]), for a pair (A, Φ) consisting of a connection on $\mathfrak{su}(E)$ and a section Φ of $W^+ \otimes E$. Equivalently, given a pair (A, Φ) with fixed-determinant connection A on E, the equations (2.1) take the same form except that F_A^+ is replaced by $(F_A^+)_0$.

In this section we briefly describe our holonomy perturbations of these equations [11]. These perturbations allow us to prove transversality for the moduli space of solutions, away from points where the connection is reducible or the spinor vanishes identically, and to prove the existence of an Uhlenbeck compactification for this perturbed moduli space.

Donaldson's proof of the connected-sum theorem for his polynomial invariants [7, Theorem B] makes use of certain 'extended anti-self-dual equations' [7, Equation (4.24)] to which the Freed-Uhlenbeck generic metrics theorem does not apply [7, §4(v)]. These extended equations model a neighborhood of the product connection appearing in the Uhlenbeck compactification of the moduli space of anti-self-dual SU(2) connections. To obtain transversality for the zero locus of the extended equations, he employs holonomy perturbations which give gauge equivariant C^∞ maps $\mathcal{A}_E^* \to \Omega^+(\mathfrak{su}(E))$ [5, §2], [7, pp. 282–287]. These perturbations are continuous with respect to Uhlenbeck limits and yield transversality not only for the top stratum,

but also for all lower strata and for all intersections of the geometric representatives defining the Donaldson invariants.

In [11, §2.4 & Appendix] we describe a generalization of Donaldson's idea which we use to prove transversality for the moduli space of solutions to a perturbed version of the PU(2) monopole equations (2.1). Unfortunately, in the case of the moduli space of PU(2) monopoles, the analysis is considerably more intricate. In Donaldson's application, some important features ensure that the requisite analysis is relatively tractable: (i) reducible connections can be excluded from the compactification of the extended moduli spaces [7, p. 283], (ii) the cohomology groups for the elliptic complex of his extended equations have simple weak semi-continuity properties with respect to Uhlenbeck limits [7, Proposition 4.33], and (iii) the zero locus being perturbed is cut out of a finite-dimensional manifold [7, p. 281, Lemma 4.35, & Corollary 4.38]. For the development of Donaldson's method for PU(2) monopoles described here and in detail in [11], none of these simplifying features hold and so the corresponding transversality argument is rather complicated. Indeed, one can see from Proposition 7.1.32 in [8] that because of the Dirac operator, the behavior of the cokernels of the linearization of the PU(2) monopole equations can be quite involved under Uhlenbeck limits. The method we describe below uses an infinite sequence of perturbing sections defined on the infinite-dimensional configuration space of pairs; when restricted to small enough open balls in the configuration space, away from reducibles, only finitely many of these perturbing sections are non-zero and they vanish along the reducibles.

Let $\mathcal{G}_E$ be the Hilbert Lie group of L^2_{k+1} unitary gauge transformations of E with *determinant one*. It is often convenient to take quotients by a slightly larger symmetry group than $\mathcal{G}_E$ when discussing pairs, so let S^1_Z denote the center of U(2) and set $^{\circ}\mathcal{G}_E := S^1_Z \times_{\{\pm \mathrm{id}_E\}} \mathcal{G}_E$, which we may view as the group of L^2_{k+1} unitary gauge transformations of E with constant determinant. The stabilizer of a unitary connection on E in $^{\circ}\mathcal{G}_E$ always contains the center $S^1_Z \subset$ U(2). We call A *irreducible* if its stabilizer is exactly S^1_Z and *reducible* otherwise. Let $\mathcal{B}_E(X) := \mathcal{A}_E(X)/\mathcal{G}_E$ be the quotient space of L^2_k connections on E with fixed determinant connection and let $\mathcal{A}^*_E(X)$ and $\mathcal{B}^*_E(X)$ be the subspace of irreducible L^2_k connections and its quotient. As before, we may equivalently view $\mathcal{B}_E(X)$ and $\mathcal{B}^*_E(X)$ as quotients of the spaces of L^2_k connections on $\mathfrak{su}(E)$ by the induced action of $\mathcal{G}_E$ on $\mathfrak{su}(E)$.

We fix $r \geq k+1$ and define gauge-equivariant C^{∞} maps, whose construction we outline below,

$$(2.2) \qquad \mathcal{A}_E(X) \ni A \mapsto \vec{\tau} \cdot \vec{\mathfrak{m}}(A) \in L^2_{k+1}(X, \mathfrak{gl}(\Lambda^+) \otimes_{\mathbb{R}} \mathfrak{so}(\mathfrak{su}(E))),$$

$$\mathcal{A}_E(X) \ni A \mapsto \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A) \in L^2_{k+1}(X, \mathrm{Hom}(W^+, W^-) \otimes_{\mathbb{C}} \mathfrak{sl}(E)),$$

where $\vec{\tau} := (\tau_{j,l,\alpha})$ is a sequence in $C^r(X, \mathfrak{gl}(\Lambda^+))$ and $\vec{\vartheta} := (\vartheta_{j,l,\alpha})$ is a sequence in $C^r(X, \Lambda^1 \otimes \mathbb{C})$, while $\vec{\mathfrak{m}}(A) := (\mathfrak{m}_{j,l,\alpha}(A))$ is a sequence in $L^2_{k+1}(X, \mathfrak{su}(E))$ of holonomy

sections, and

$$\vec{\tau} \cdot \vec{\mathfrak{m}}(A) := \sum_{j,l,\alpha} \tau_{j,l,\alpha} \otimes_{\mathbb{R}} \mathrm{ad}(\mathfrak{m}_{j,l,\alpha}(A)),$$

$$\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A) := \sum_{j,l,\alpha} \rho(\vartheta_{j,l,\alpha}) \otimes_{\mathbb{C}} \mathfrak{m}_{j,l,\alpha}(A).$$

We digress briefly to outline the construction of the gauge equivariant maps $\mathfrak{m}_{j,l,\alpha}$.

Let $\gamma \subset X$ be a C^∞ loop based at a point $x_0 \in X$ and let $\mathrm{Hol}_{\gamma,x_0}(A) \in \mathrm{SO}(\mathfrak{su}(E))|_{x_0}$ be the holonomy of a smooth $\mathrm{SO}(3)$ connection A around the loop γ. The exponential map $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$ gives a diffeomorphism from a ball $2D$ around the origin in $\mathfrak{so}(3)$ to a ball around the identity in $\mathrm{SO}(3)$. Let $\psi : \mathbb{R} \to [0,1]$ be a C^∞ cutoff function such that $\psi(|\zeta|) = 1$ for $\zeta \in \frac{1}{2}D$, $\psi(|\zeta|) > 0$ for $\zeta \in D$, and $\psi(|\zeta|) = 0$ for $\zeta \in \mathfrak{so}(3) - D$ and define

$$\mathfrak{hol}_{\gamma,x_0}(A) := \psi\left(|\exp^{-1}(\mathrm{Hol}_{\gamma,x_0}(A))|\right) \cdot \mathrm{ad}^{-1} \exp^{-1}(\mathrm{Hol}_{\gamma,x_0}(A)).$$

Suppose $Y \subset X$ is a simply-connected open subset containing x_0. If $A|_Y$ is irreducible then there are three loops $\{\gamma_l\}_{l=1}^3 \subset Y$, depending on A, such that the set $\{\mathfrak{hol}_{\gamma_l,x_0}(A)\}_{l=1}^3$ is a basis for $\mathfrak{su}(E)|_{x_0} = \mathrm{ad}^{-1}(\mathfrak{so}(\mathfrak{su}(E)|_{x_0}))$. We extend $\mathfrak{hol}_{\gamma,x_0}(A)$ to a C^∞ section $\hat{\mathfrak{h}}_\gamma(A)$ of $\mathfrak{su}(E)$ by radial parallel translation, with respect to A over a small ball $2B$ and then multiplying by a C^∞ cutoff function φ on X which is positive on B and identically zero on $X - B$. The set $\{\varphi\hat{\mathfrak{h}}_{\gamma_l}(A)|_y\}_{l=1}^3$ then spans $\mathfrak{su}(E)|_y$ for $y \in B$.

For an L_k^2 unitary connection A on E with $k \geq 2$, the section $\hat{\mathfrak{h}}_\gamma(A)$ need not be in L_{k+1}^2. So we use the Neumann heat operator , for fixed small $t > 0$, to construct an L_{k+1}^2 section

$$\mathfrak{h}_\gamma(A) := \exp(-t d_{A|2B}^* d_{A|2B})\hat{\mathfrak{h}}_\gamma(A)$$

of $\mathfrak{su}(E)$ over $2B$ which converges to $\hat{\mathfrak{h}}_\gamma(A)$ in $C^0(B)$ as $t \to 0$. Therefore, for a small enough time $t(A)$, the set $\{\varphi\mathfrak{h}_{\gamma_l}(A)|_y\}_{l=1}^3$, spans $\mathfrak{su}(E)|_y$ for all $y \in B$ just as before. If A_0 is an L_k^2 unitary connection on $E|_{2B}$ and $\gamma_1, \gamma_2, \gamma_3$ are loops in $2B$ based at x_0 such that $\{\mathfrak{h}_{\gamma_l,x_0}(A_0)\}_{l=1}^3$ spans $\mathfrak{su}(E)|_{x_0}$, then there is a positive constant $\varepsilon(A_0, \{\gamma_l\})$ such that if A is an L_k^2 unitary connection on $E|_{2B}$ with $\|A - A_0\|_{L_{k,A_0}^2(2B)} < \varepsilon$, then the set $\{\mathfrak{h}_{\gamma_l,x_0}(A)\}_{l=1}^3$ spans $\mathfrak{su}(E)|_B$.

Given this digression, we can now construct the maps $\mathfrak{m}_{j,l,\alpha}$. Let $\{4B_j\}_{j=1}^{N_b}$, be a disjoint collection of open balls with centers x_j and radius R_0, where N_b is a fixed integer to be determined later. The quotient space $\mathcal{B}_E^*(2B_j)$ of irreducible L_k^2 unitary connections on $E|_{2B_j}$ is a paracompact C^∞ manifold modelled on a separable Hilbert space. For each $j = 1, \dots, N_b$ and each point $[A_0]$ in $\mathcal{B}_E^*(2B_j)$, we can find loops $\{\gamma_{j,l,A_0}\}_{l=1}^3$, contained in $2B_j$ and based at x_j such that $\{\mathfrak{h}_{\gamma_{j,l,A_0}}(A_0)\}_{l=1}^3$, spans $\mathfrak{u}(E)|_{B_j}$. For each such point $[A_0]$, there is an L_{k,A_0}^2 ball $B_{[A_0]}(\varepsilon_{A_0}) \subset \mathcal{B}_E^*(2B_j)$ such that for all $[A] \in B_{[A_0]}(\varepsilon_{A_0})$, the set $\{\mathfrak{h}_{\gamma_{j,l,A_0}}(A)\}_{l=1}^3$ spans $\mathfrak{su}(E)|_{B_j}$. These balls give

an open cover of $\mathcal{B}_E^*(2B_j)$ and hence there is a locally finite refinement of this open cover, $\{U_{j,\alpha}\}_{\alpha=1}^\infty$, and a collection of suitably chosen C^∞ cutoff functions $\chi_{j,\alpha}$ with supports in the L_k^2 balls containing the $U_{j,\alpha}$ such that

$$\sum_\alpha \chi_{j,\alpha}[A] > 0, \qquad [A] \in \mathcal{B}_E^*(2B_j), \quad j = 1, \ldots, N_b.$$

Hence, for each $U_{j,\alpha}$, we obtain loops $\{\gamma_{j,l,\alpha}\}_{l=1}^3 \subset 2B_j$ such that for all $[A] \in U_{j,\alpha}$, the sections $\mathfrak{h}_{\gamma_{j,l,\alpha}}(A)$ span $\mathfrak{su}(E)|_{B_j}$. The cutoff functions $\chi_{j,\alpha}$ are chosen so that all their derivatives are bounded (in the obvious sense analogous to (2.5) below).

Let β be a C^∞ cutoff function on $\mathbb{R}$ such that $\beta(t) = 1$ for $t \le \frac{1}{2}$ and $\beta(t) = 0$ for $t \ge 1$, with $\beta(t) > 0$ for $t < 1$. Then the C^∞ gauge-invariant maps $\mathcal{A}_E(X) \to \mathbb{R}$, $A \mapsto \beta_j[A]$, given by

$$(2.3) \qquad \beta_j[A] := \beta \left(\frac{1}{\varepsilon_0^2} \int_{B(x_j, 4R_0)} \beta \left(\frac{\operatorname{dist}(\cdot, x_j)}{4R_0} \right) |F_A|^2 \, dV \right)$$

are zero when the energy of the connection A is greater than or equal to $\frac{1}{2}\varepsilon_0^2$ over a ball $2B_j$. Here, $\varepsilon_0(g, A_{\det W^+}, A_{\det E})$ is a universal constant [11]. Finally, we define C^∞ cutoff functions on X by setting $\varphi_j := \beta(\operatorname{dist}_g(\cdot, x_j)/R_0)$, so that φ_j is positive on the ball B_j and zero on its complement in X. We define gauge-equivariant C^∞ maps $\mathcal{A}_E(X) \to L_{k+1}^2(X, \mathfrak{su}(E))$, $A \mapsto \mathfrak{m}_{j,l,\alpha}(A)$, by setting

$$(2.4) \qquad \mathfrak{m}_{j,l,\alpha}(A) := \beta_j[A]\chi_{j,\alpha}\left([A|_{2B_j}]\right) \varphi_j \mathfrak{h}_{\gamma_{j,l,\alpha}}(A).$$

Thus, at each point $A \in \mathcal{A}_E(X)$ only a finite number of the $\mathfrak{m}_{j,l,\alpha}(A)$ are non-zero and we show that each map $\mathfrak{m}_{j,l,\alpha}$ is C^∞ with uniformly bounded derivatives of all orders on $\mathcal{A}_E(X)$ in the sense of (2.5). The energy cutoff functions β_j ensure continuity across the Uhlenbeck boundary. The universal bound on $\|F_A\|_{L^2(X)}^2$ in (3.2) below for solutions (A, Φ) to the perturbed PU(2) monopole equations (2.7) ensures that the number N_b of balls B_j may be chosen sufficiently large that for every PU(2) monopole (A, Φ), there is at least one ball $B_{j'}$ whose associated holonomy sections $\{\mathfrak{m}_{j',l,\alpha}(A)\}_{l=1}^3$ span $\mathfrak{su}(E)|_{B_{j'}}$.

The parameters $\vec{\tau}$ and $\vec{\vartheta}$ vary in the Banach spaces of $\ell_\delta^1(\mathbb{A})$ sequences in $C^r(X, \mathfrak{gl}(\Lambda^+))$ and $C^r(X, \Lambda^1 \otimes \mathbb{C})$, respectively, where $\mathbb{A} = \{(j, l, \alpha)\}$, with norm

$$\|\vec{\vartheta}\|_{\ell_\delta^1(C^r(X))} := \sum_{j,l,\alpha} \delta_\alpha^{-1} \|\vartheta_{j,l,\alpha}\|_{C^r(X)},$$

and similarly for $\|\vec{\tau}\|_{\ell_\delta^1(C^r(X))}$. For any open subset $U \subset \mathcal{A}_E(X)$ and C^s map $\mathfrak{f} : U \to L_{k+1}^2(X, \mathfrak{su}(E))$, we define

$$(2.5) \qquad \|\mathfrak{f}\|_{C^s(U)} := \sup_{\substack{A \in U}} \sup_{\substack{1 \le i \le s \\ \|a_i\|_{L_{k,A}^2(X)} \le 1}} \|(D^s\mathfrak{f})_A(a_1, \ldots, a_s)\|_{L_{k+1,A}^2(X)},$$

and similarly for maps to $L^2_{k+1}(X, \mathfrak{gl}(\Lambda^+) \otimes \mathfrak{so}(\mathfrak{su}(E)))$ and $L^2_{k+1}(X, \mathrm{Hom}(W^+, W^-) \otimes \mathfrak{sl}(E))$. The C^s norm of maps on $\mathcal{A}_E(X)$ defined by (2.5) is gauge invariant and so descends to a C^s norm on sections over the quotient $\mathcal{B}^*_E(X)$.

Proposition 2.1. [11] *There exists a sequence* $\delta := (\delta_\alpha)^\infty_{\alpha=1} \in \ell^\infty((0,1])$ *of positive weights such that the gauge-equivariant maps in* (2.2) *are* C^∞, *with uniformly bounded differentials of all orders,*

$$\|\vec{\vartheta} \cdot \vec{\mathfrak{m}}\|_{C^s(\mathcal{A}_E)} \le C\|\vec{\vartheta}\|_{\ell^1_\delta(C^r(X))} \quad and \quad \|\vec{\tau} \cdot \vec{\mathfrak{m}}\|_{C^s(\mathcal{A}_E)} \le C\|\vec{\tau}\|_{\ell^1_\delta(C^r(X))},$$

for all $s \ge 0$ *and positive constants* $C = C(g, k, s)$.

The parameters $\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}$ can be chosen so that

$$(2.6) \qquad \|\tau_0\|_{L^\infty(X)} + \sup_{A \in \mathcal{A}_E} \|\vec{\tau} \cdot \vec{\mathfrak{m}}(A)\|_{L^\infty(X)} < \tfrac{1}{64},$$

$$\|\vartheta_0\|_{L^\infty_1(X)} \sup_{A \in \mathcal{A}_E} \|\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)\|_{L^\infty_{1,A}(X)} \le 1,$$

provided $k \ge 3$ and we shall therefore assume this constraint is in effect for the remainder of the article. These bounds are used in §3 and they in turn follow from the C^0 estimates of Proposition 2.1 for

$$\|\vec{\tau}\|_{\ell^1_\delta(C^r(X))} \le \varepsilon_\tau \quad and \quad \|\vec{\vartheta}\|_{\ell^1_\delta(C^r(X))} \le \varepsilon_\vartheta,$$

with suitable constants ε_τ and ε_ϑ when $s = 0$.

We call an L^2_k pair (A, Φ) in the *pre-configuration space*,

$$\tilde{\mathcal{C}}_{W,E} := \mathcal{A}_E \times L^2_k(X, W^+ \otimes E),$$

a PU(2) *monopole* if $\mathfrak{S}(A, \Phi) = 0$, where the $^\circ\mathcal{G}$ equivariant map $\mathfrak{S} : \tilde{\mathcal{C}}_{W,E} \to L^2_k(\Lambda^+ \otimes \mathfrak{su}(E)) \oplus L^2_k(W^- \otimes E)$ is defined by

$$(2.7) \qquad \mathfrak{S}(A, \Phi) := \begin{pmatrix} F^+_A - (\mathrm{id} + \tau_0 \otimes \mathrm{id}_{\mathfrak{su}(E)} + \vec{\tau} \cdot \vec{\mathfrak{m}}(A))(\rho^+)^{-1}(\Phi \otimes \Phi^*)_{00} \\ D_A\Phi + \rho(\vartheta_0)\Phi + \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)\Phi \end{pmatrix}.$$

We let $M_{W,E} := \mathfrak{S}^{-1}(0)$ be the moduli space of solutions cut out of the *configuration space*,

$$\mathcal{C}_{W,E} := \tilde{\mathcal{C}}_{W,E}/^\circ\mathcal{G}_E,$$

by the section (2.7), where $u \in {}^\circ\mathcal{G}_E$ acts by $u(A, \Phi) := (u_*A, u\Phi)$. We note that the perturbations in (2.7) are *zero-order*, unlike the first-order perturbations considered [35]. They differ from those in [27] as they are not 'central in E', that is, at some points $x \in X$, they span $\mathfrak{su}(E)|_x$.

We let $\mathcal{C}^{*,0}_{W,E} \subset \mathcal{C}_{W,E}$ be the subspace of pairs $[A, \Phi]$ such that A is irreducible and the section Φ is not identically zero and set $M^{*,0}_{W,E} = M_{W,E} \cap \mathcal{C}^{*,0}_{W,E}$. Note that we have a canonical inclusion $\mathcal{B}_E \subset \mathcal{C}_{W,E}$ given by $[A] \mapsto [A, 0]$ and similarly for the pre-configuration spaces.

The sections $\vec{\tau} \cdot \vec{\mathfrak{m}}(A)$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)$ vanish at reducible connections A by construction; plainly, the terms in the PU(2) monopole equations (2.7) involving the perturbations $\vec{\tau} \cdot \vec{\mathfrak{m}}(A)$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)$ are zero when Φ is zero. The holonomy perturbations considered by Donaldson in [7] are inhomogeneous, as he uses the perturbations to kill the cokernels of d_A^+ directly. In contrast, the perturbations we consider in (2.7) are homogeneous and we argue indirectly that the cokernels of the linearization vanish away from the reducibles and zero-section solutions.

3. Uhlenbeck compactness

We describe the Uhlenbeck closure of the moduli space of PU(2) monopoles and outline the proof of compactness.

3.1. **Statement of main Uhlenbeck compactness results.** We say that a sequence of points $[A_\beta, \Phi_\beta]$ in $\mathcal{C}_{W,E}$ *converges* to a point $[A, \Phi, \mathbf{x}]$ in $\mathcal{C}_{W,E_{-\ell}} \times \mathrm{Sym}^\ell(X)$, where $E_{-\ell}$ is a Hermitian two-plane bundle over X such that

$$\det(E_{-\ell}) = \det E \quad \text{and} \quad c_2(E_{-\ell}) = c_2(E) - \ell, \quad \text{with} \quad \ell \in \mathbb{Z}_{\geq 0},$$

if the following hold:

- There is a sequence of $L^2_{k+1,\mathrm{loc}}$ determinant-one, unitary bundle isomorphisms $u_\beta : E|_{X \setminus \mathbf{x}} \to E_{-\ell}|_{X \setminus \mathbf{x}}$ such that the sequence of monopoles $u_\beta(A_\beta, \Phi_\beta)$ converges to (A, Φ) in $L^2_{k,\mathrm{loc}}$ over $X \setminus \mathbf{x}$, and
- The sequence of measures $|F_{A_\beta}|^2$ converges in the weak-* topology on measures to $|F_A|^2 + 8\pi^2 \sum_{x \in \mathbf{x}} \delta(x)$.

The holonomy-perturbation maps in (2.2) are continuous with respect to Uhlenbeck limits, just as are those of [7]. Suppose $\{A_\beta\}$ is a sequence in $\mathcal{A}_E(X)$ which converges to an Uhlenbeck limit $(A, \mathbf{x})$ in $\mathcal{A}_{E_{-\ell}}(X) \times \mathrm{Sym}^\ell(X)$. The sections $\vec{\tau} \cdot \vec{\mathfrak{m}}(A_\beta)$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A_\beta)$ then converge in $L^2_{k+1}(X)$ to a section $\vec{\tau} \cdot \vec{\mathfrak{m}}(A, \mathbf{x})$ of $\mathfrak{gl}(\Lambda^+) \otimes \mathfrak{so}(\mathfrak{su}(E_{-\ell}))$ and a section $\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A, \mathbf{x})$ of $\mathrm{Hom}(W^+, W^-) \otimes \mathfrak{sl}(E_{-\ell})$, respectively. For each $\ell \geq 0$, the maps of (2.2) extend continuously to gauge-equivariant maps

$$(3.1) \qquad \mathcal{A}_{E_{-\ell}}(X) \times \mathrm{Sym}^\ell(X) \to L^2_{k+1}(X, \mathfrak{gl}(\Lambda^+) \otimes_{\mathbb{R}} \mathfrak{so}(\mathfrak{su}(E_{-\ell}))),$$
$$\mathcal{A}_{E_{-\ell}}(X) \times \mathrm{Sym}^\ell(X) \to L^2_{k+1}(X, \mathrm{Hom}(W^+, W^-) \otimes_{\mathbb{C}} \mathfrak{sl}(E_{-\ell})),$$

given by $(A, \mathbf{x}) \mapsto \vec{\tau} \cdot \vec{\mathfrak{m}}(A, \mathbf{x})$ and $(A, \mathbf{x}) \mapsto \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A, \mathbf{x})$, which are C^∞ on each C^∞ stratum determined by $\mathrm{Sym}^\ell(X)$.

Our construction of the Uhlenbeck compactification for $M_{W,E}$ requires us to consider moduli spaces

$$\mathbf{M}_{W,E_{-\ell}} \subset \mathcal{C}_{W,E_{-\ell}} \times \mathrm{Sym}^\ell(X)$$

of triples $[A, \Phi, \mathbf{x}]$ given by the zero locus of the $^\circ\mathcal{G}_{E_{-\ell}}$ equivariant map

$$\mathfrak{S} : \mathcal{C}_{W,E_{-\ell}} \times \mathrm{Sym}^\ell(X) \to L^2_k(\Lambda^+ \otimes \mathfrak{su}(E_{-\ell})) \oplus L^2_k(W^- \otimes E_{-\ell})$$

defined as in (2.7) except using the perturbing sections $\vec{\tau} \cdot \vec{\mathfrak{m}}$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}$ in (3.1) instead of those in (2.2). We call $\mathbf{M}_{W,E_{-\ell}}$ a *lower-level* moduli space if $\ell > 0$ and call $\mathbf{M}_{W,E_{-0}} = M_{W,E}$ the *top* or *highest level*.

In the more familiar case of the unperturbed PU(2) monopole equations (2.1), the spaces $\mathbf{M}_{W,E_{-\ell}}$ would simply be products $M_{W,E_{-\ell}} \times \mathrm{Sym}^\ell(X)$. In general, though, the spaces $\mathbf{M}_{W,E_{-\ell}}$ are not products when $\ell > 0$ due to the slight dependence of the section $\mathfrak{S}(A, \Phi, \mathbf{x})$ on the points $\mathbf{x} \in \mathrm{Sym}^\ell(X)$ through the perturbations $\vec{\tau} \cdot \vec{\mathfrak{m}}$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}$. A similar phenomenon is encountered in [7, §4(iv)–(vi)] for the case of the extended anti-self-dual equations, where holonomy perturbations are also employed in order to achieve transversality.

We define $\bar{M}_{W,E}$ to be the Uhlenbeck closure of $M_{W,E}$ in the space of ideal PU(2) monopoles,

$$IM_{W,E} := \bigcup_{\ell=0}^{N} \mathbf{M}_{W,E_{-\ell}} \subset \bigcup_{\ell=0}^{N} \left(\mathcal{C}_{W,E_{-\ell}} \times \mathrm{Sym}^\ell(X) \right)$$

for any integer $N \geq N_p$, where N_p is a sufficiently large constant to be specified below.

Theorem 3.1. [11] *Let X be a closed, oriented, smooth four-manifold with C^∞ Riemannian metric, spin^c structure (ρ, W) with spin^c connection, and a Hermitian two-plane bundle E with unitary connection on $\det E$. Then there is a positive integer N_p, depending at most on the curvatures of the fixed connections on W and $\det E$ together with $c_2(E)$, such that for all $N \geq N_p$ the topological space $\bar{M}_{W,E}$ is second countable, Hausdorff, compact, and given by the closure of $M_{W,E}$ in $\cup_{\ell=0}^{N} \mathbf{M}_{W,E_{-\ell}}$.*

Remark 3.2. The existence of an Uhlenbeck compactification for the moduli space of solutions to the unperturbed PU(2) monopole equations (2.1) was announced by Pidstrigach [25] and an argument was outlined in [27]. A similar argument for the equations (2.1) was outlined by Okonek and Teleman in [24]. Theorem 3.1 yields the standard Uhlenbeck compactification for the system (2.1) and for the perturbations of (2.1) described in [9, 36] — see Remark 4.3. An independent proof of Uhlenbeck compactness for (2.1) and certain perturbations of these equations is given in [35, 36].

3.2. **Outline of the proof of Theorem 3.1.** The proofs of existence of an Uhlenbeck compactification for the moduli spaces of solutions to (2.1) and (2.7) proceed along similar lines. The common thread is the use of Bochner-Weitzenböck formulas to give bounds on the 'energy' and on the self-dual component of the curvature

$$(3.2) \qquad \int_X \left(|F_A|^2 + |\Phi|^4 + |\nabla_A \Phi|^2 \right) dV \leq M < \infty,$$

$$(3.3) \qquad \|F_A^+\|_{C^0(X)} \leq K,$$

for any point $[A, \Phi] \in M_{W,E}$, with constant K depending only the data in the hypotheses of Theorem 3.1. Granted the bounds (3.2) and (3.3), which are not obvious in the presence of infinite sequences of holonomy perturbations, the proof of Theorem 3.1

then follows the example of [8, pp. 163–165], [17, Theorem III.3.2] (due to Taubes), and [33, Proposition 4.4] for the moduli space of anti-self-dual connections. The scale invariance of the PU(2) monopole equations is used in much the same way that the conformal invariance of the anti-self-dual equation is exploited in [8]: if (A, Φ) is a solution to (2.7) for a Riemannian metric g, then $(A, \lambda\Phi)$ is a solution for the metric $\lambda^{-2}g$.

The first step is to prove C^∞ regularity of L_1^2 solutions to the PU(2) monopole equations (2.7) coupled with a Coulomb gauge-fixing condition with C^∞ reference pair when $(\vec{\tau}, \vec{\vartheta}) = (0, 0)$ and C^∞ regularity of L_2^2 solutions for any C^∞ perturbations $(\vec{\tau}, \vec{\vartheta})$. We next record a generalization of the usual Bochner-Weitzenböck identity for the Dirac operator on $\Omega^0(W^+)$ (see [18, 22, 40]) to the Dirac operator on $\Omega^0(W^+ \otimes E)$:

$$(3.4) \qquad D_A^* D_A = \nabla_A^* \nabla_A + \tfrac{1}{4}R + \rho^+(F_A^+) + \tfrac{1}{2}\rho^+(F^+(A_{\det E}) + F^+(A_{\det W^+})),$$

$$(3.5) \qquad D_A D_A^* = \nabla_A^* \nabla_A + \tfrac{1}{4}R + \rho^-(F_A^-) + \tfrac{1}{2}\rho^-(F^+(A_{\det E}) + F^-(A_{\det W^+})),$$

where R is the scalar curvature of the metric g. As in the case of the U(1) monopole equations [18, 40], the Bochner-Weitzenböck identity (3.4), integration by parts, the estimates on the holonomy perturbations given by (2.6), and the maximum principle applied to $|\Phi|^2$, yield *a priori* estimates for Φ and F_A when (A, Φ) is a solution to the PU(2) monopole equations (2.7):

Lemma 3.3. *There is a positive constant K, depending only on the data in the hypotheses of Theorem 3.1, such that the following holds. If (A, Φ) is an L_1^2 solution to* (2.7)*, then*

$$(3.6) \qquad \|\Phi\|_{L^4(X)} + \|\nabla_A \Phi\|_{L^2(X)} \le K \quad and \quad \|F_A\|_{L^2(X)} \le K.$$

If (A, Φ) is a C^1 solution to (2.7)*, then*

$$(3.7) \qquad \|\Phi\|_{C^0(X)} \le K \quad and \quad \|F_A^+\|_{C^0(X)} \le K.$$

Remark 3.4. A modification of the vanishing argument in [40] shows that the only solutions to (2.7) on S^4, with its standard metric, are of the form $(A, 0)$ with A anti-self-dual.

The bounds (3.2) and (3.3) are thus given by Lemma 3.3. The energy bound (3.2) together with the C^∞ regularity of L_1^2 solutions to (2.7) and a local Coulomb gauge-fixing condition are then used to prove removability of point singularities for PU(2) monopoles when $(\vec{\tau}, \vec{\vartheta}) = 0$ following [37, 39]. One only needs removability of singularities in this case as, by construction, the holonomy perturbations vanish near points of concentrated curvature. The proof of Theorem 3.1 is now completed by the usual measure-theoretic arguments, together with the local Coulomb gauge-fixing theorem and the patching arguments of [38]. One can either use the Chern-Simons function, as in [8, §4.4], to show that the singular points have integer multiplicities or use the cutting-off procedure of [33].

4. Transversality

In this section we explain why, for generic perturbation parameters $(\tau_0, \vec{\tau}, \vec{\vartheta})$, the moduli space $M_{W,E}$ of solutions to the perturbed PU(2) monopole equations (2.7) is a smooth manifold on the complement of the subspaces of zero-section and reducible solutions.

4.1. Statement of main transversality results.

The space $\mathrm{Sym}^{\ell}(X)$ is smoothly stratified, the strata being enumerated by partitions of ℓ. If $\Sigma \subset \mathrm{Sym}^{\ell}(X)$ is a smooth stratum, we define

$$\mathbf{M}_{W,E_{-\ell}}|_{\Sigma} := \{[A, \Phi, \mathbf{x}] \in \mathbf{M}_{W,E_{-\ell}} : \mathbf{x} \in \Sigma\},$$

with $\mathbf{M}_{W,E_{-0}} := M_{W,E}$ when $\ell = 0$.

Theorem 4.1. [11] *Let X be a closed, oriented, smooth four-manifold with C^{∞} Riemannian metric, spin^c structure (ρ, W) with spin^c connection, and a Hermitian line bundle $\det E$ with unitary connection. Then there exists a first-category subset of the space of C^{∞} perturbation parameters such that the following holds: For each 4-tuple $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ in the complement of this first-category subset, integer $\ell \geq 0$, and smooth stratum $\Sigma \subset \mathrm{Sym}^{\ell}(X)$, the moduli space $\mathbf{M}_{W,E_{-\ell}}^{*,0}(\Sigma; \tau_0, \vec{\tau}, \vec{\vartheta})$ is a smooth manifold of the expected dimension*

$$\begin{aligned}
\dim \mathbf{M}_{W,E_{-\ell}}^{*,0}|_{\Sigma} &= \dim M_{W,E_{-\ell}}^{*,0} + \dim \Sigma \\
&= -2p_1(\mathfrak{su}(E_{-\ell})) - \tfrac{3}{2}(e(X) + \sigma(X)) + \dim \Sigma \\
&\quad + \tfrac{1}{2}p_1(\mathfrak{su}(E_{-\ell})) + \tfrac{1}{2}((c_1(W^+) + c_1(E))^2 - \sigma(X)) - 1,
\end{aligned}$$

where $c_1(E_{-\ell}) = c_1(E)$ and $c_2(E_{-\ell}) = c_2(E) - \ell$.

Remark 4.2.   1. Over the complement of a first-category subset in Σ, comprising the regular values of the projection maps onto the second factor Σ, the projection $\mathbf{M}_{W,E_{-\ell}}^{*,0}|_{\Sigma} \to \Sigma$ is a smooth fiber bundle with fibers $\mathbf{M}_{W,E_{-\ell}}^{*,0}|_{\mathbf{x}}$, for $\mathbf{x} \in \Sigma$. Indeed, the only tangent vectors in each stratum Σ which might not appear in the image of the projection are those arising from the radial vector on the annuli $4B_j - 2\bar{B}_j$. This observation shows that the projection from $\mathbf{M}_{W,E_{-\ell}}^{*,0}|_{\Sigma}$ to Σ is transverse to certain submanifolds of Σ which allows dimension-counting arguments [12]. An approach to dimension counting in the presence of holonomy perturbations is also discussed by Donaldson in [7, pp. 282–287].

2. Although not required by Theorem 4.1, a choice of generic Riemannian metric on X ensures that the moduli space $M_E^{\mathrm{asd},*}$ of irreducible anti-self-dual connections on $\mathfrak{su}(E)$ is smooth and of the expected dimension [8, 16], although the points of $M_E^{\mathrm{asd},*}$ need not be regular points of $M_{W,E}^*$ as the linearization of (2.7) need not be surjective [12].

3. A choice of generic parameter τ_0 ensures that the moduli spaces $M_{W,E,L_1}^{\mathrm{red},0}$ of non-zero-section solutions to (2.7) which are reducible with respect to the splitting $E = L_1 \oplus (\det E) \otimes L_1^*$ are smooth and of the expected dimension [12], although the points of $M_{W,E,L_1}^{\mathrm{red},0}$ need not be regular points of $M_{W,E}^0$ since the linearization of (2.7) need not be surjective, as we see in [12].

Remark 4.3. Different approaches to the question of transversality for the PU(2) monopole equations (2.1) with generic perturbation parameters have been considered by Pidstrigach and Tyurin in [27] and by Teleman in [35]; see [11] for further details. More recently, a new approach to transversality for (2.1) has been discovered independently by the first author [9] and by Teleman [36]: the method uses only the perturbations (τ_0, ϑ_0) together with perturbations of the Riemannian metric on X and compatible Clifford map. Although these perturbations are simpler than those of Theorem 4.1, the proofs of transversality are lengthy and difficult.

While the description of the holonomy perturbations outlined above may appear fairly complicated at first glance, in practice they do not present any major difficulties beyond those that would be encountered if simpler perturbations not involving the bundle $\mathfrak{su}(E)$ (such as the Riemannian metric on X or the connection on $\det W^+$) were sufficient to achieve transversality [12, 13, 14]. We note that related transversality and compactness issues have been recently considered in approaches to defining Gromov-Witten invariants for general symplectic manifolds [20, 28, 30].

4.2. **Outline of the proof of Theorem 4.1.** We shall outline the proof when $\ell = 0$ and indicate the very small modification required when $\ell > 0$. Let $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_\tau \oplus \mathcal{P}_\vartheta$ denote our Banach space of C^r perturbation parameters. Define a $^\circ\mathcal{G}_E$-equivariant map

$$\underline{\mathfrak{S}} : \mathcal{P} \times \tilde{\mathcal{C}}_{W,E} \to L_{k-1}^2(\Lambda^+ \otimes \mathfrak{su}(E)) \oplus L_{k-1}^2(W^- \otimes E)$$

by setting

$$(4.1) \quad \underline{\mathfrak{S}}(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}, A, \Phi) := \begin{pmatrix} F_A^+ - (\mathrm{id} + \tau_0 \otimes \mathrm{id}_{\mathfrak{su}(E)} + \vec{\tau} \cdot \vec{\mathfrak{m}}(A)) \rho^{-1}(\Phi \otimes \Phi^*)_{00} \\ D_A \Phi + \rho(\vartheta_0) \Phi + \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A) \Phi \end{pmatrix},$$

where $^\circ\mathcal{G}_E$ acts trivially on the space of perturbations $\mathcal{P}$, and so the parametrized moduli space $\mathfrak{M}_{W,E} := \underline{\mathfrak{S}}^{-1}(0)/^\circ\mathcal{G}_E$ is a subset of $\mathcal{P} \times \mathcal{C}_{W,E}$. We let $\mathfrak{M}_{W,E}^{*,0} := \mathfrak{M}_{W,E} \cap (\mathcal{P} \times \mathcal{C}_{W,E}^{*,0})$. The $^\circ\mathcal{G}_E$-equivariant map $\underline{\mathfrak{S}}$ defines, in the usual way, a section of a Banach vector bundle $\underline{\mathfrak{V}}$ over $\mathcal{P} \times \mathcal{C}_{W,E}^{*,0}$, and a Fredholm section $\mathfrak{S} := \underline{\mathfrak{S}}(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}, \cdot)$ of the Banach vector bundle $\mathfrak{V} := \underline{\mathfrak{V}}|_{(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})}$ over $\{\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}\} \times \mathcal{C}_{W,E}^{*,0}$ for each point $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}) \in \mathcal{P}$. The principal goal, then, is to prove

Theorem 4.4. [11] *The zero set in $\mathcal{P} \times \mathcal{C}_{W,E}^{*,0}$ of the section $\underline{\mathfrak{S}}$ is regular and, in particular, the moduli space $\mathfrak{M}_{W,E}^{*,0}$ is a smooth Banach submanifold of $\mathcal{P} \times \mathcal{C}_{W,E}^{*,0}$.*

Accepting Theorem 4.4 for the moment, the Sard-Smale theorem [31] (in the form of Proposition 4.3.11 in [8]) implies that the zero sets in $\mathcal{C}_{W,E}^{*,0}$ of the sections $\mathfrak{S}_{\tau_0,\vartheta_0,\vec{\tau},\vec{\vartheta}}$ are regular for all C^r perturbations $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ in the complement in $\mathcal{P}$ of a first-category subset and so Theorem 4.1 is a standard consequence of Theorem 4.4 for C^r parameters. The use of perturbation parameters which are only known to be C^r, while necessary to apply the Sard-Smale theorem, proves inconvenient in practice. The result can be sharpened, however, so that only C^∞ parameters are needed by adapting a similar argument due to Taubes for the Seiberg-Witten moduli space (see [29]). This gives Theorem 4.1 when $\ell = 0$. The proof of Theorem 4.1 when $\ell > 0$ is the same as for the case $\ell = 0$, except that the $^\circ\mathcal{G}_E$ equivariant map $\underline{\mathfrak{S}}$ is now defined on $\mathcal{P} \times \tilde{\mathcal{C}}_{W,E-\ell} \times \Sigma$, where Σ is a smooth stratum of $\mathrm{Sym}^\ell(X)$.

If $[\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}, A, \Phi]$ is a point in $\underline{\mathfrak{S}}^{-1}(0)$, elliptic regularity for the equations (2.7) ensures that the point $[A, \Phi]$ has a C^r representative (A, Φ). Similarly, if (v, ψ) is an L_{k-1}^2 element of the cokernel of $D\underline{\mathfrak{S}}$ at the point $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}, A, \Phi)$, elliptic regularity for the Laplacian $D\underline{\mathfrak{S}}(D\underline{\mathfrak{S}})^*$, with C^{r-1} coefficients, implies that (v, ψ) is in C^{r+1}. From the definition of the holonomy-perturbation sections $\vec{\tau} \cdot \vec{\mathfrak{m}}(A)$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)$, the linearization of (4.1), and some linear algebra we obtain

Proposition 4.5. [11] *Suppose* $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}, A, \Phi)$ *is a point in* $\underline{\mathfrak{S}}^{-1}(0)$ *with A irreducible and* $\Phi \not\equiv 0$. *If (v, ψ) is in the cokernel of* $D\underline{\mathfrak{S}}$ *at* $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}, A, \Phi)$ *then* $(v, \psi) \equiv 0$ *on each ball B_j whose holonomy sections* $\{\mathfrak{m}_{j,l,\alpha}(A)\}$ *span* $\mathfrak{su}(E)|_{B_j}$.

In order that the holonomy sections $\{\mathfrak{m}_{j,l,\alpha}(A)\}$ span $\mathfrak{su}(E)|_{B_j}$, we see from the construction of §2 that the connection $A|_{2B_j}$ must be irreducible. We call $U \subset X$ an *admissible open set for a connection A* if it contains all closed balls $\bar{B}_j$ for which $\beta_j[A] > 0$. (The supports of all the sections $\mathfrak{m}_{j,l,\alpha}(A)$ are contained in $\cup_{j=1}^{N_b} \bar{B}_j$ and so any open subset of X containing $\cup_{j=1}^{N_b} \bar{B}_j$ is admissible.) Thus, in order to bring Proposition 4.5 to bear on the proof of Theorem 4.4, we must know that if (A, Φ) is an irreducible, non-zero-section solution to (2.7) on X, then the restriction of A to an admissible open set is irreducible. This crucial property is implied by the following unique continuation result for reducible PU(2) monopoles:

Theorem 4.6. [11] *If (A, Φ) is a C^r solution of the perturbed* PU(2) *monopole equations* (2.7) *with $\Phi \not\equiv 0$ over a connected, oriented, smooth four-manifold X with C^r Riemannian metric and (A, Φ) is reducible on a non-empty open subset $U \subset X$ with $\bar{B}_j \subset U$ for all j such that $\beta_j[A] > 0$, then (A, Φ) is reducible on X.*

The corresponding property for anti-self-dual connections is proved as Lemma 4.3.21 in [8]. The proof of Donaldson and Kronheimer relies on the Agmon-Nirenberg unique continuation theorem for an ordinary differential inequality on a Hilbert space [2, Theorem 2]. We show in [11] that their proof adapts to the case of the PU(2) monopole equations (2.1) or (2.7), when the initial open set where (A, Φ) is reducible contains the closed balls $\bar{B}(x_j, R_0)$ supporting holonomy perturbations. The proof of

Theorem 4.6 has two main steps. The first and more difficult one is a local extension result for stabilizers of pairs which are reducible on a suitable ball:

Proposition 4.7. [11] *Let X be an oriented, smooth four-manifold with C^r Riemannian metric and injectivity radius $\varrho = \varrho(x_0)$ at a point x_0. Suppose that $0 < r_0 < r_1 \leq \frac{1}{2}\varrho$. Let (A, Φ) be a C^r pair solving the $\mathrm{PU}(2)$ monopole equations (2.7) on X. If u is a C^{r+1} gauge transformation of $E|_{B(x_0,r_0)}$ satisfying $u(A, \Phi) = (A, \Phi)$ on $B(x_0, r_0)$, and if either $B_j \cap B(x_0, r_1) = \emptyset$ or $B_j \subset B(x_0, r_0)$, for all balls B_j for which $\beta_j[A] > 0$, then there is an extension of u to a C^{r+1} gauge transformation $\hat{u}$ of $E|_{B(x_0,r_1)}$ with $\hat{u}(A, \Phi) = (A, \Phi)$ on $B(x_0, r_1)$.*

Proposition 4.7 is a consequence (see [1]) of the Agmon-Nirenberg unique continuation theorem for an ordinary differential inequality on a Hilbert space [2, Theorem 2(ii)], together with the observation that the holonomy perturbations vanish outside the balls B_j with $\beta_j[A] > 0$ and on any ball B_j with $\beta_j[A] = 0$, so the PU(2) monopole equations are local with respect to the pair (A, Φ) on the open subset of X where the stabilizer is to be extended. The second and much shorter step is to complete the proof of Theorem 4.6 by showing that the local stabilizers, produced by repeatedly applying Proposition 4.7, fit together to give a non-trivial stabilizer and thus a reducible pair over the whole manifold X.

Granted Proposition 4.5 and Theorem 4.6, the proof of Theorem 4.4 is easily concluded. Let $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}, A, \Phi)$ be a C^r representative for a point in $\mathfrak{M}_{W,E}^{*,0}$, so that A is irreducible and $\Phi \not\equiv 0$, and suppose (v, ψ) is in the cokernel of $D\mathfrak{S}$ at the point $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}, A, \Phi)$. By definition of N_b we have $\beta_j[A] > 0$ for at least one index $j \in \{1, \ldots, N_b\}$. Since A is irreducible on X, Theorem 4.6 implies that $A|_{B(x_{j'}, 2R_0)}$ must be irreducible for some $j' \in \{1, \ldots, N_b\}$ such that $\beta_{j'}[A] > 0$; otherwise, $A|_{B(x_{j'}, 2R_0)}$ would be reducible for all j such that $\beta_j[A] > 0$ and Theorem 4.6 would imply that A would be reducible over all of X, contradicting our assumption that A is irreducible. Hence, there is at least one ball $B_{j'}$ whose holonomy sections $\{\mathfrak{m}_{j',l,\alpha}(A)\}$ span $\mathfrak{su}(E)|_{B_{j'}}$. Now Proposition 4.5 implies that $(v, \psi) \equiv 0$ on the set $\bar{B}_J(A)$ of all balls $\bar{B}_j$ for which $\beta_j[A] > 0$ and $A|_{2B_j}$ irreducible. Such elements of the cokernel of $D\mathfrak{S}$ have the unique continuation property by the Aronszajn-Cordes unique continuation theorem for second-order elliptic inequalities with a real scalar principal symbol [3]. Indeed, the Laplacian $D\mathfrak{S}(D\mathfrak{S})^*$ is a differential operator on $X - \bar{B}_J(A)$ and the coefficients $\delta\mathfrak{m}_{j,l,\alpha}/\delta A$ containing non-local, integral-operator terms obtained by differentiating holonomies with respect to A (see [11, §A.2] or [34, §8]) are supported on $B_J(A)$, precisely where $(v, \psi) \equiv 0$. Thus, (v, ψ) may be viewed as a C^{r+1} element of the kernel of the second-order, elliptic, differential operator $D\mathfrak{S}^0(D\mathfrak{S}^0)^*$ with C^{r-1} coefficients over X obtained by setting all holonomy perturbations and their derivatives equal to zero. (Without loss of generality, we may scale the Dirac equation in (2.7) by $1/\sqrt{2}$, so the symbol of $D\mathfrak{S}^0(D\mathfrak{S}^0)^*$ is $\frac{1}{2}$ times the

Riemannian metric on T^*X.) Hence, $(v, \psi) \equiv 0$ on X and this completes the proof of Theorem 4.4.

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